Multi-Period Performance Attribution: Framework for an Allocation Effect Taking Active Weight Drift into Account

We have investigated the behavior of the Brinson model when used for evaluating the outperformance over multiple periods. We have shown that the allocation effect calculated over multiple periods can capture next to the added value of allocation decisions an effect that arises due to the drift in weights introduced by selection decisions. By an extension to the Brinson method this effect can be isolated resulting in a more intuitive attribution analysis. Furthermore we have evaluated different smoothing algorithms.

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Introduction

In investment management the main method for ex-post evaluation of the decisions that are made in portfolio construction and portfolio management is the Brinson attribution model developed by Brinson et al. [1985,1986]. This model is widely used in the market and allows for analyzing the added value of decisions that are made in the investment decision process.

A challenging aspect of the Brinson model is that it is a single period model, using the start of period weights and the return over the period to explain the difference in return between the portfolio and benchmark over that specific period. A natural choice for the period of the analysis would be to fully cover the time between two sequential decisions.

In practice the outcome of the Brinson analysis is required whenever a stakeholder wants to monitor the investment decisions. Often weekly, monthly or quarterly reports are sent out, without taking into account the exact moment that decisions were made. Most reports cover several sub periods with different sequential decisions. Reports should be available at any moment over any period, regardless of the length of the period over which investment decisions were made. The solution would be to shorten the period over which you perform the analysis and combine the outcomes of multiple analyses. If you choose to combine the results of multiple analyses you will also encounter the situation that you will combine results for periods in which no decisions are taken.

In this article we will describe challenges that arise when combining the attribution effects of the Brinson model over multiple periods with special attention for the drift of the weights during periods in which no decisions are taken. We will show that the standard multi-period Brinson framework will have spill-over effects. Part of the result of selection decisions will show up as a result of the allocation decisions, even when no allocation decision is taken.

This effect was already observed by Ryan [2001]. He presented a framework that separated the contribution of active allocation decisions from passive allocation decisions (drift) by the introduction of return neutralized weights. We will introduce a different framework that better isolates this effect in two new measures, the drift-allocation and the drift-interaction. The results of this multi-period framework will be more in line with the results when the period was analyzed as being a single period.

We will start the article with an overview of the Brinson model, followed by a section on the challenges that are tackled by existing literature on multi-period Brinson attribution. Next we will show that the current framework for multi-period Brinson attribution provides some unintuitive results. We will discuss why these results arise and provide a framework for more intuitive results for multi-period Brinson attribution. We will also apply this framework to a number of cases.

Brinson model

Brinson et al. [1985,1986] introduced a model with which you can attribute the difference between the return of a portfolio and the return of a benchmark, also called the active or excess return, to contributions coming from allocation and selection decisions. The model assumes that every active investment decision is a decision between or inside one or more segments. The weighted sum of the returns of the segments gives the return of the portfolio and benchmark. There are two types of decisions that can be made that impact the return of the portfolio compared to its benchmark. The first would be to over- or underweight a segment compared to the weight of the benchmark, also known as an allocation decision. The second would be to select a different investment for a segment with a different return compared to that segment's benchmark, also known as a selection decision.

Exhibit 1

	Portfolio Segment Returns	Benchmark Segment Returns
Portfolio Segment Weights	(Q4) Portfolio $\sum_j w_j^P * r_j^P$	(Q2) Active Asset Allocation Portfolio $\sum_j w_j^P * r_j^B$
Benchmark Segment Weights	(Q3) Active Stock Selection Portfolio $\sum_j w_j^B * r_j^P$	(Q1) Benchmark $\sum_j w_j^B * r_j^B$

Notional Portfolios as described by Brinson et al. [1986]

Exhibit 1 shows the Brinson model that is used for understanding the sources of active performance. The bottom right quadrant (Q1) shows the benchmark return as being the weighted sum of the benchmark segment returns using the benchmark weights. The fourth quadrant (Q4) shows a similar calculation to obtain the portfolio return.

The top right quadrant (Q2) shows the return that would be realized with the portfolio weights, but invested inside the segments in accordance with the benchmark. A comparison of the return of the top right quadrant (Q2) with the bottom right quadrant (Q1) shows the impact of the allocation decisions that are made. The formula for the overall allocation effect is:

$$\text{allocation effect} = \sum_j w_j^P * r_j^B - \sum_j w_j^B * r_j^B = \sum_j \left(w_j^P - w_j^B\right) * r_j^B \quad (1)$$

where the sum is taken over all segments. In this equation w_j^p is the weight of the j^{th} segment in the portfolio, w_j^B the weight of the j^{th} segment in the benchmark and r_j^B the benchmark return of the j^{th} segment. The formula above is suboptimal for the allocation effect for the individual segments since overweighting a segment with a negative return can be a good decision if all other segments performed even worse. Therefore the preferred way to calculate the allocation effect in the case that an allocation decision in one segment needs to be offset by an allocation in other segments is:

$$\text{allocation effect} = \sum_j \left(w_j^P - w_j^B\right) * \left(r_j^B - R_{TOT}^B\right) \quad (2)$$

where R_{TOT}^B is the overall return of the benchmark. The total allocation effect is unchanged since the sum of $\sum_j \left(w_j^P - w_j^B\right)$ equals zero and therefore $\sum_j \left(w_j^P - w_j^B\right) * R_{TOT}^B$ equals zero. In the remainder of this article we will use formula 2 for the allocation effect.

The bottom left quadrant (Q3) of *Exhibit 1* shows what return would have been realized by a reference portfolio when it would be invested inside the segments according to the portfolio but was held according to the weights of the segments in the benchmark. A comparison of the bottom left quadrant (Q3) with the bottom right quadrant (Q1) shows the impact of the selection decisions that are made. The selection effect can be written as

$$\text{selection effect} = \sum_j w_j^B * \left(r_j^P - r_j^B\right) \tag{3}$$

where r_j^p is the portfolio return of the j[th] segment.

The allocation and selection effect do not add up to the difference between the portfolio and benchmark return. To explain the full difference an interaction effect is present that captures the combined effect that will occur in the case that both a selection and an allocation decision is made. It will be positive for segments for which the portfolio return outperforms the benchmark and that have a higher allocation in the portfolio compared to the benchmark or for segments where the portfolio return underperforms against the benchmark and which are also underweighted. The formula for the interaction effect is

$$\text{interaction effect} = \sum_j \left(w_j^P - w_j^B\right) * (r_j^P - r_j^B) \tag{4}$$

The Brinson model is an arithmetic model. The sum of the allocation, selection and interaction effects add up to the difference of the portfolio and benchmark return. This holds for one period as can be seen in *Exhibit 2a*. In this exhibit we present the attribution analysis for two subsequent periods. The manager has the investment choice between two segments and decides to overweight segment 1 and consequently underweight segment 2 compared to the benchmark weights. He also decides to select a manager with an active mandate and thereby deviate from the benchmark for segment 1 and therefore the portfolio return for segment 1 differs from the benchmark return. For segment 2 he decides to stay passive and invest according to the benchmark. Consequently the portfolio and benchmark returns are the same.

The benchmark return of segment 1 was higher than the total benchmark return. Since we used formula 2 for the allocation effect in the analysis, the overweighting of segment 1 was a good decision in the first period. As a consequence the underweighting of segment 2 was equally good since the benchmark was invested half in segment 1 and half in segment 2. The total benchmark return lies precisely in between the benchmark returns of the two segments and therefore the absolute difference between the benchmark return of segment 1 and the total benchmark return is the same as the absolute difference between the benchmark return of segment 2 and the total benchmark return. Only the sign is opposite. This, together with the fact that the amount with which segment 1 is overweighed is equal to the amount with which sector 2 is underweighted, leads to both segments having the same allocation effect.

Exhibit 2

		Portfolio Weight	Benchmark Weight	Portfolio Return	Benchmark Return	Active Return	Allocation effect	Selection Effect	Interaction Effect	Total Effect
a)	**Period 1**									
	Segment 1	60.0%	50.0%	12.00%	8.00%	4.00%	0.65%	2.00%	0.40%	3.05%
	Segment 2	40.0%	50.0%	-5.00%	-5.00%	0.00%	0.65%	0.00%	0.00%	0.65%
	Total	**100.0%**	**100.0%**	**5.20%**	**1.50%**	**3.70%**	**1.30%**	**2.00%**	**0.40%**	**3.70%**
b)	**Period 2**									
	Segment 1	63.9%	53.2%	-3.00%	-2.00%	-1.00%	-0.20%	-0.53%	-0.11%	-0.84%
	Segment 2	36.1%	46.8%	2.00%	2.00%	0.00%	-0.23%	0.00%	0.00%	-0.23%
	Total	**100.0%**	**100.0%**	**-1.19%**	**-0.13%**	**-1.07%**	**-0.43%**	**-0.53%**	**-0.11%**	**-1.07%**
c)	**Two period as sum of single periods results**									
	Segment 1	60.0%	50.0%	8.64%	5.84%	2.80%	0.45%	1.46%	0.29%	2.20%
	Segment 2	40.0%	50.0%	-3.10%	-3.10%	0.00%	0.42%	0.00%	0.00%	0.42%
	Total	**100.0%**	**100.0%**	**3.94%**	**1.37%**	**2.57%**	**0.87%**	**1.46%**	**0.29%**	**2.62%**
d)	**Two period as a single period**									
	Segment 1	60.0%	50.0%	8.64%	5.84%	2.80%	0.45%	1.40%	0.28%	2.13%
	Segment 2	40.0%	50.0%	-3.10%	-3.10%	0.00%	0.45%	0.00%	0.00%	0.45%
	Total	**100.0%**	**100.0%**	**3.94%**	**1.37%**	**2.57%**	**0.89%**	**1.40%**	**0.28%**	**2.57%**

Example of an Brinson Attribution analysis where formulas 2-4 are used. a) and b) show two single period Brinson Analyses. c) shows an analysis where the results from a) and b) are summed and d) shows a Brinson analysis where the two periods are combined before the analysis is performed.

The selection effect in the first period for segment 1 is positive since the portfolio did outperform the benchmark. Since segment 1 had an effect arising from both the allocation and selection decisions, also an interaction effect occurs in both periods for segment 1. For segment 2 the selection effect is zero since no selection bet was taken for this segment, and consequently the interaction effect is zero.

Two period Brinson attribution for two segments

In the previous section we have explained the Brinson model for two segments. In this section we will investigate combining the results for two subsequent periods. In *Exhibit 2b* we have included a second period in which no additional decisions are made. In the second period we assumed that the portfolio of the first segment underperformed against the benchmark and therefore the selection effect is negative.

In *Exhibit 2c* we also calculated the two period effects by adding the single period effects geometrically. Adding the effects together does, however, not explain the excess return over the two periods, although both single period analyses do explain the full excess return of the corresponding period. If you would do a separate analysis for the two periods acting as one period, as is done in *Exhibit 2d*, then the analysis again explains the full excess return.

The difference between the total effect for the two periods and the aggregation over the two periods as shown in *Exhibit 2c* is caused by the fact that the attribution effects are arithmetic effects, explaining the arithmetic difference between the portfolio and benchmark return. For the two periods the portfolio and benchmark returns are linked geometrically and a new two-period arithmetic excess return is calculated. In cumulating both period's portfolio and benchmark return, the second period return will also be generated over the added value of the first period. This additional compounding effect is not explained by the attribution effects of the first period or those of the second period, and therefore adding the attribution effects will not explain the full two period excess return.

Smoothing Algorithms

A number of methods have been developed that solve the problem of the attribution effects no longer adding up to the total excess return when the attribution results of multiple periods are combined. The methods can roughly be separated into two categories. The first category cumulates the results of the 4 quadrant portfolios over time and calculates the attribution effects based on the cumulated portfolios. Examples of these methods are the Compounded Notional Portfolios (CNP) method developed by Davies and Laker [2001] , David [2006] and Berg [2014]. The second category uses algorithms to adjust the single period attribution effects such that they do sum up over multiple periods to exactly explain the total added value. Those algorithms are also known as smoothing algorithms. Examples of this category have been created by Cariño [1999], Menchero [2000] and Frongello [2002].

In *Exhibit 3* we present the attribution results for the two period analyses using the methods developed by Cariño and Berg. The methods developed by Menchero and Frongello produce very similar results as the method developed by Cariño for this analysis as we will show in *Exhibit 4*. As can be seen the methods developed by Berg and Cariño (*Exhibit 3 b and c*) do give slightly different attribution results, especially for the segment selection and interaction effects. They also both differ from the case where we analyze the two periods as if they were one period and using the original one period Brinson model (*Exhibit 3d*).

No selection decision is made for segment 2, therefore the selection effect for this segment in the method of Berg is unintuitive. This selection effect arises since the Notional Portfolios used for the calculation of the attribution effects of the segments are corrected with the return of the notional portfolio in the previous period. Segment 1 has a selection effect in the first period and therefore the return of the total selection notional portfolio will have a return in the first period. This will affect the notional portfolio of segment 2 for the second period resulting in a selection effect for segment 2 in the analysis. In the appendix we show that this unintuitive selection effect is present due to the nature of the CNP methodology and will be present for any CNP method.

Exhibit 3

		Allocation effect	Selection Effect	Interaction Effect	Total Effect
a)	**Two periods as sum of single periods results**				
	Segment 1	0.45%	1.46%	0.29%	2.20%
	Segment 2	0.42%	0.00%	0.00%	0.42%
	Total	**0.87%**	**1.46%**	**0.29%**	**2.62%**
b)	**Two periods using Cariño smoothing method**				
	Segment 1	0.44%	1.44%	0.29%	2.16%
	Segment 2	0.41%	0.00%	0.00%	0.41%
	Total	**0.85%**	**1.44%**	**0.29%**	**2.57%**
c)	**Two Periods using Berg CNP method**				
	Segment 1	0.43%	1.43%	0.27%	2.13%
	Segment 2	0.43%	0.02%	0.00%	0.45%
	Total	**0.86%**	**1.45%**	**0.27%**	**2.57%**
d)	**Two periods as a single period**				
	Segment 1	0.45%	1.40%	0.28%	2.13%
	Segment 2	0.45%	0.00%	0.00%	0.45%
	Total	**0.89%**	**1.40%**	**0.28%**	**2.57%**

Continuation of the example of Exhibit 2. a) shows the same attribution effects of the two period in the case that the single period effects are added as is presented in Exhibit 2c. b) shows the attribution effects in the case that the Cariño smoothing algorithm is applied. c) shows the attribution effects when applying the method using the method developed by Berg. In d) the effects are shown if the two periods are first combined before the analysis is performed (the same as Exhibit 2d).

In *Exhibit 3* we have presented analyses in which we have generated the returns artificially. This can lead to speculation whether the effects that we have presented are specific for the returns that were chosen. Furthermore we have only presented the results for two periods. To get a better feeling of how the presented framework will hold over multiple periods using real investment data, we have created an equity portfolio that consists of both developed market and emerging market stocks. The benchmark for this portfolio consists of 50% MSCI developed markets and 50% MSCI emerging markets[1]. The manager decides to make a selection effect and excludes Australia from the developed markets and includes the China A shares in the emerging markets. The decision is implemented by selecting passive funds that invest exactly conform the index and therefore have the same return as the index. An allocation decision is made to overweight the developed market segment with 5% and consequently underweight the emerging market segment with 5%.

[1] MSCI Developed markets data from:
http://www.msci.com/products/indexes/country_and_regional/dm/performance.html
MSCI Emerging market data from:
http://www.msci.com/products/indexes/country_and_regional/em/performance.html

Exhibit 4

		Excess Eeturn	Allocation Effect	Selection Effect	Interaction Effect
a)	**Single period analysis**				
	Developed Markets		0.73%	0.44%	0.04%
	Emerging Markets		0.73%	0.31%	-0.03%
	Total	**2.23%**	**1.46%**	**0.75%**	**0.01%**
b)	**Daily analysis, Cariño smoothed**				
	Developed Markets		0.68%	0.43%	0.04%
	Emerging Markets		0.76%	0.38%	-0.05%
	Total	**2.23%**	**1.43%**	**0.81%**	**-0.02%**
c)	**Daily analysis, Menchero smoothed**				
	Developed Markets		0.67%	0.43%	0.04%
	Emerging Markets		0.75%	0.40%	-0.06%
	Total	**2.23%**	**1.42%**	**0.82%**	**-0.02%**
d)	**Daily analysis, Frongello smoothed**				
	Developed Markets		0.68%	0.43%	0.04%
	Emerging Markets		0.76%	0.38%	-0.05%
	Total	**2.23%**	**1.43%**	**0.81%**	**-0.01%**

			Active Allocation Effect	Passive Allocation Effect	Selection Effect	Interaction Effect
e)	**Daily analyses, using Ryan's model**					
	Developed Markets		0.70%	-0.03%	0.43%	0.04%
	Emerging Markets		0.78%	-0.03%	0.38%	-0.05%
	Total	**2.23%**	**1.48%**	**-0.05%**	**0.81%**	**-0.02%**

Attribution analysis for a portfolio over 2013 containing Developed and Emerging market investments. The Benchmark is invested 50% in the MSCI Developed markets and 50 % in the MSCI Emerging market. The portfolio is invested 55% in the MSCI Developed Markets excluding Australia and 45% in the MSCI Emerging market including China A Share. a) presents the single period analysis where the return over the whole year is used. b) until d) presents daily analysis that are combined to get yearly results. Three methods to smooth the results such that the analysis explains the full return are presented. b) has used the Cariño smoothing algorithm, c) has used the smoothing algorithm developed by Menchero and d) is smoothed using the Frongello smoothing algorithm. e) shows the results if you would apply the model developed by Ryan in which the Cariño smoothing algorithm is applied. We have used the net USD returns.

In *Exhibit 4* four different methods to explain the result over 2013 are presented. In a) we show the single period Brinson analysis. In b) we have calculated the Brinson effects on a daily basis and combined the results using the algorithm developed by Cariño. In c) the daily effects are combined using Menchero's smoothing algorithm and d) presents the results using Frongello's method. The results of the attribution analyses using the smoothing algorithms of

Cariño, Menchero and Frongello are very much the same. In the remainder of this article we will use the smoothing algorithm developed by Cariño.

Drift Allocation Effect

When using the Brinson attribution analysis for multiple periods additional methods are needed to sum the segment effects up to the total added value, as the analysis presented in *Exhibit 2 and 3* shows. Those methods do sometimes result in unintuitive results like the selection effect for a segment where no selection decision is made in the case of CNP methods. A way to evaluate models with multiple degrees of freedom is by setting one degree to zero and evaluate the outcome of the model for the remaining degrees of freedom. Within the Brinson framework there are two degrees of freedom, the allocation decisions and the selection decisions. To investigate the smoothed Brinson method further for other unintuitive results we have altered the case presented in *Exhibit 2 and 3* by removing the allocation decision. The portfolio and benchmark weights are the same and therefore the portfolio is invested half in segment 1 and half in segment 2. The attribution results for this case are presented in *Exhibit 5*.

Since no allocation decision is made during the two periods we would expect that there is no allocation effect for any of the segments. However, when we look at the outcome in *Exhibit 5c*, we see an allocation effect arising for both segments. The cause of this effect lies within the Brinson model and is not due to the smoothing algorithm that is used. At the start of the first period the weight for all segments, both in the portfolio and in the benchmark, is 50% and, therefore, there is no allocation effect in the first period. In the first period the portfolio has a different return compared to the benchmark return for segment 1 which results in different portfolio and benchmark weights at the end of the first period for the individual segments. For the attribution analysis of the second period the weights at the end of the first period are used in the Brinson attribution model and since the end of the first period portfolio and benchmark weights are drifted apart, an allocation effect arises for the second period. This effect arises independent of the smoothing method that is used to create multi-period attribution effects that explain the excess return over multiple periods.

This effect was also observed by Ryan [2001]. He introduces return neutralized weights to be used in the allocation effect calculation to separate the allocation effect coming from active allocation decisions from the portfolio drift and passive decisions. The return neutralized weights are effectively the start of the previous period weights. We will show that it is more natural to use passive portfolio weights to measure the contribution of the allocation decisions.

We will now investigate in which cases the weights will drift within a period. The weight at the end of the period for a segment j can be written as:

$$w_j^e = \frac{w_j^b * (1 + r_j)}{\sum_{i=1}^{n} w_i^b * (1 + r_i)} = \frac{w_j^b * (1 + r_j)}{(1 + R_{TOT})} \tag{5}$$

In this equation w_j^e is the weight of the jth segment at the end of the period, w_j^b is the weight of the jth segment at the start of the period, r_j is the return of the segment during the period and R_{TOT} is the overall return during the period. The excess weight between the portfolio and benchmark at the end of a period can be written as:

$$w_j^{P,e} - w_j^{B,e} = \frac{w_j^{P,b} * \left(1 + r_j^P\right)}{1 + R_{TOT}^P} - \frac{w_j^{B,b} * \left(1 + r_j^B\right)}{1 + R_{TOT}^B} \tag{6}$$

From this equation it can be seen that a neutral allocation position at the start of a period will drift to an active weight differences at the end of the period when at least one segment has a selection result. Exceptions are the segments that have no weight at all, when only one segment contains all investments and when a segment has the same ratio between the segments portfolio return and the overall portfolio return as the ratio between the segments benchmark return and the overall benchmark return.

Exhibit 5

		Portfolio Weight	Benchmark Weight	Portfolio Return	Benchmark Return	Active Return	Allocation effect	Selection Effect	Interaction Effect	Total Effect
a)	**Period 1**									
	Segment 1	50.0%	50.0%	12.00%	8.00%	4.00%	0.00%	2.00%	0.00%	2.00%
	Segment 2	50.0%	50.0%	-5.00%	-5.00%	0.00%	0.00%	0.00%	0.00%	0.00%
	Total	**100.0%**	**100.0%**	**3.50%**	**1.50%**	**2.00%**	**0.00%**	**2.00%**	**0.00%**	**2.00%**
b)	**Period 2**									
	Segment 1	54.1%	53.2%	-3.00%	-2.00%	-1.00%	-0.02%	-0.53%	-0.01%	-0.56%
	Segment 2	45.9%	46.8%	2.00%	2.00%	0.00%	-0.02%	0.00%	0.00%	-0.02%
	Total	**100.0%**	**100.0%**	**-0.71%**	**-0.13%**	**-0.58%**	**-0.04%**	**-0.53%**	**-0.01%**	**-0.58%**
c)	**Two periods using Cariño smoothing method**									
	Segment 1	50.0%	50.0%	8.64%	5.84%	2.80%	-0.02%	1.45%	-0.01%	1.42%
	Segment 2	50.0%	50.0%	-3.10%	-3.10%	0.00%	-0.02%	0.00%	0.00%	-0.02%
	Total	**100.0%**	**100.0%**	2.77%	1.37%	**1.40%**	**-0.04%**	**1.45%**	**-0.01%**	**1.40%**
d)	**Two periods as a single period**									
	Segment 1	50.0%	50.0%	8.64%	5.84%	2.80%	0.00%	1.40%	0.00%	1.40%
	Segment 2	50.0%	50.0%	-3.10%	-3.10%	0.00%	0.00%	0.00%	0.00%	0.00%
	Total	**100.0%**	**100.0%**	2.77%	1.37%	**1.40%**	**0.00%**	**1.40%**	**0.00%**	**1.40%**

Attribution example in the case that no allocation decision is made. a) and b) show two single period analysis. c) shows the combined two period effect when using the Carino smoothing algorithm. d) shows the result in the case that the analysis is performed as a single period.

To adjust the Brinson model for this allocation effect arising from selection decisions in prior periods, we have to introduce a new weight which is the weight of the portfolio in the case that no selection decisions were taken. We will call this the passive portfolio (PP) weight since it represents the weight that the portfolio would have had if it was invested passively (according to the benchmark) inside all the segments. The passive portfolio weight at the start of the first period is the same as the portfolio weight. The weight for segment j at the end of the first period is given by the following equation:

$$w_j^{PP,e} = {w_j^{P,b} * (1 + r_j^B)} \Big/ {(1 + R_{TOT}^{B,PP})} \tag{7}$$

Where $R_{TOT}^{B,PP}$ is the return that you get when you sum the benchmark return times the start of period passive portfolio weight of all the segments. The passive portfolio weights should be set equal to the portfolio weights at the moment that a rebalancing of the portfolio takes place or a new allocation decision is made.

We can adjust the Brinson model by introducing a new effect which captures this introduced allocation effect that is due to the effect of the selection decisions in prior periods. Since this effect captures the part of the allocation effect which is not due to the real allocation decision but is caused by the weights drifting due to the selection decisions we will call it the drift-allocation effect.

$$\text{Drift} - \text{Allocation effect} = \sum_j \left(w_j^P - w_j^{PP}\right) * (r_j^B - R_{TOT}^B) \quad (8)$$

The allocation effect then also needs to be adjusted accordingly:

$$\text{Allocation effect} = \sum_j \left(w_j^{PP} - w_j^B\right) * (r_j^B - R_{TOT}^B) \quad (9)$$

Please note that this adjustment is only an adjustment in the case of an analysis over multiple periods. For a one period analysis $w_j^{PP} = w_j^P$ holds and the allocation effect again reduces to the standard Brinson attribution equation (2).

In *Exhibit 5b* it can be seen that there is an interaction effect introduced which is due to the interaction of the drift-allocation with the selection. The Brinson model can also be adjusted by introducing a drift-interaction effect that captures this effect.

$$\text{Drift} - \text{Interation effect} = \sum_j \left(w_j^P - w_j^{PP}\right) * (r_j^P - r_j^B) \quad (10)$$

$$\text{Interation effect} = \sum_j \left(w_j^{PP} - w_j^B\right) * (r_j^P - r_j^B) \quad (11)$$

The drift-allocation effect will only explain a part of the excess contribution for a segment of the second period in the case that there is an excess return for the segment in the first period and the benchmark return of the segment differs from the overall benchmark return in the second period.

Ryan [2001] proposes to use return neutralized weights for both the portfolio and the benchmark to separate the allocation effect due to active management from other passive effects. Effectively he proposes to use the weights at the start of the previous period or at the moment that the allocation decision was made to measure the impact of the active allocation decision instead of the weights at the start of the period over which you are measuring the added value. You would get the same allocation effect in the case that you would perform a periodic rebalancing of both the portfolio and the benchmark at the start of each period to the weights at the start of the previous period. Therefore the active allocation proposed by Ryan incorporates apart from the allocation effect that is corrected for the drift allocation as proposed in formula 9 a rebalancing term. In *Exhibit 4e)* we have applied the model of Ryan. As you can see the Active Allocation Effects are different from the results from the single period analyses as presented in *Exhibit 4a).*

Exhibit 6

		Allocation effect	Drift-Allocation Effect	Selection Effect	Interaction Effect	Drift-Interaction Effect	Total Effect
a)	**Period 1**						
	Segment 1	0.00%	0.00%	2.00%	0.00%	0.00%	2.00%
	Segment 2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
	Total	**0.00%**	**0.00%**	**2.00%**	**0.00%**	**0.00%**	**2.00%**
b)	**Period 2**						
	Segment 1	0.00%	-0.02%	-0.53%	0.00%	-0.01%	-0.53%
	Segment 2	0.00%	-0.02%	0.00%	0.00%	0.00%	0.00%
	Total	**0.00%**	**-0.04%**	**-0.53%**	**0.00%**	**-0.01%**	**-0.58%**
c)	**Two period using Cariño smoothing method**						
	Segment 1	0.00%	-0.02%	1.45%	0.00%	-0.01%	1.42%
	Segment 2	0.00%	-0.02%	0.00%	0.00%	0.00%	-0.02%
	Total	**0.00%**	**-0.04%**	**1.45%**	**0.00%**	**-0.01%**	**1.40%**
d)	**Two period as a single period**						
	Segment 1	0.00%	0.00%	1.40%	0.00%	0.00%	1.40%
	Segment 2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
	Total	**0.00%**	**0.00%**	**1.40%**	**0.00%**	**0.00%**	**1.40%**

Same analysis as in Exhibit 5 but with the Drift-Allocation Effect and Drift-Interaction Effect.

In *Exhibit 6* we have included the drift-allocation effect, the drift-interaction effect and the altered formulas for the allocation and interaction effect. *Exhibit 6* presents the results for the case that no allocation decision is made and, as expected, the allocation effect and interaction effect are now zero. The drift-allocation effect and drift-interaction effect are exactly the same as the allocation effect and the interaction effect of the original effects in *Exhibit 5* which was to be expected. In *Exhibit 7* we have taken the original case (from *Exhibit 2*) where there is an allocation decision and included the drift-allocation and drift-interaction effect.

Exhibit 7

		Allocation effect	Drift-Allocation Effect	Selection Effect	Interaction Effect	Drift-Interaction Effect	Total Effect
a)	**Period 1**						
	Segment 1	0.65%	0.00%	2.00%	0.40%	0.00%	3.05%
	Segment 2	0.65%	0.00%	0.00%	0.00%	0.00%	0.65%
	Total	**1.30%**	**0.00%**	**2.00%**	**0.40%**	**0.00%**	**3.70%**
b)	**Period 2**						
	Segment 1	-0.18%	-0.02%	-0.53%	-0.10%	-0.01%	-0.81%
	Segment 2	-0.21%	-0.02%	0.00%	0.00%	0.00%	-0.21%
	Total	**-0.39%**	**-0.03%**	**-0.53%**	**-0.10%**	**-0.01%**	**-1.07%**
c)	**Two period using Cariño smoothing method**						
	Segment 1	0.46%	-0.02%	1.44%	0.30%	-0.01%	2.16%
	Segment 2	0.43%	-0.02%	0.00%	0.00%	0.00%	0.41%
	Total	**0.88%**	**-0.03%**	**1.44%**	**0.30%**	**-0.01%**	**2.57%**
d)	**Two period as a single period**						
	Segment 1	0.45%	0.00%	1.40%	0.28%	0.00%	2.13%
	Segment 2	0.45%	0.00%	0.00%	0.00%	0.00%	0.45%
	Total	**0.89%**	**0.00%**	**1.40%**	**0.28%**	**0.00%**	**2.57%**

Same analysis as in Exhibit 2 but with the Drift-Allocation Effect and Drift-Interaction

When we compare the two-period results in *Exhibit 7c* with the single period result (*Exhibit 7d*), there is one effect that we have not yet explained, and that is that the allocation effect for both segments is not the same while they are the same in the single period analysis. Since the benchmark weights for the two segments are equal, the decision to overweight one segment automatically determines the underweighting for the other segment. Therefore you would expect to have equal allocation effects for the two segments. This is the case in the single period analysis, but not for the multi-period examples.

The cause for the unexpected difference in the allocation effect between the two sectors lies in the total benchmark return that is used in the equation for the allocation effect. In the first period the weight of both segments is equal and, therefore, the total return lies exactly in between the returns of the two segments. The active weight between the segments is the same in absolute value, but the sign is different. That results in the fact that both parts of the allocation equation (equation 9) are the same for the two segments and the sign is opposite for both parts, resulting in equal allocation effects for the two segments.

Exhibit 8

a) **Period 1**	**Allocation effect**	**Drift-Allocation Effect**	**Selection Effect**	**Interaction Effect**	**Drift-Interaction Effect**	**Total Effect**
Segment 1	0.65%	0.00%	2.00%	0.40%	0.00%	3.05%
Segment 2	0.65%	0.00%	0.00%	0.00%	0.00%	0.65%
Total	**1.30%**	**0.00%**	**2.00%**	**0.40%**	**0.00%**	**3.70%**
b) **Period 2**						
Segment 1	-0.20%	-0.02%	-0.53%	-0.10%	-0.01%	-0.83%
Segment 2	-0.20%	-0.02%	0.00%	0.00%	0.00%	-0.20%
Total	**-0.39%**	**-0.03%**	**-0.53%**	**-0.10%**	**-0.01%**	**-1.07%**
c) **Two periods using Cariño smoothing method**						
Segment 1	0.44%	-0.02%	1.44%	0.30%	-0.01%	2.18%
Segment 2	0.44%	-0.02%	0.00%	0.00%	0.00%	0.44%
Total	**0.88%**	**-0.03%**	**1.44%**	**0.30%**	**-0.01%**	**2.57%**
d) **Two periods as a single period**						
Segment 1	0.45%	0.00%	1.40%	0.28%	0.00%	2.13%
Segment 2	0.45%	0.00%	0.00%	0.00%	0.00%	0.45%
Total	**0.89%**	**0.00%**	**1.40%**	**0.28%**	**0.00%**	**2.57%**

The same analysis as in Exhibit 7 but we apply a periodically rebalanced total benchmark return in the allocation effect instead of the total benchmark return.

The difference of the benchmark returns of the two segments in the first period will cause benchmark weights at the end of the first period to be changed compared to the weights at the start of the first period. Because of this the total benchmark return for the second period does not lie exactly between the two segments returns anymore, but will be closer to the return of the segment that had the highest return of the two segments in the first period. In the equation for the allocation, the size of the active weight will still be equal for the two segments but with opposite signs. The second part of the equation, where the total benchmark return is subtracted from the segment return, will not have the same size anymore. This explains the difference between the allocation effects for the two segments for the second period.

If the two allocation effects of the two periods are aggregated, the multi-period allocation effect will be different from the one-period allocation effect. It will not only differ in size, but also in the comparison of which segment contributed more to the overall excess return. This is regardless of the smoothing algorithm that is used. We have now demonstrated the effect by using only two segments with equal start weights. For this situation the expectation is clear. But the same effect also occurs when you perform an analysis where the weights are not equal or if you perform an analysis over more than two segments. For those cases it is more difficult to determine the impact of this effect since the expected allocation effect for the different segments is not the same.

This effect can be removed from the allocation effect by using a periodically rebalanced total benchmark return instead of the total benchmark return in the allocation effect. The formula for the allocation effect will then become:

$$\text{Allocation effect} = \sum_j \left(w_j^{PP} - w_j^{B}\right) * (r_j^B - R_{TOT}^{PRB}) \tag{12}$$

where R_{TOT}^{PRB} is the periodically rebalanced total benchmark return. The substitution of the total benchmark return with the periodically rebalanced total benchmark return will not affect the allocation effect for the parent segment as we have shown before, but will only shift some of the allocation effect from one segment to another. When a periodically rebalanced total benchmark return is used in the allocation equation, the size between the return of the segments and the periodically rebalanced total benchmark return is equal again but opposite of sign, resulting in equal allocation effects for the two segments. The same can be applied to the drift-allocation effect which then would become:

$$\text{Drift} - \text{Allocation effect} = \sum_j \left(w_j^{P} - w_j^{PP}\right) * (r_j^B - R_{TOT}^{PRB}) \tag{13}$$

In *Exhibit 8* we show the attribution results in the case that we apply a periodically rebalanced total benchmark return in the allocation and drift-allocation effect for the example in *Exhibit 2* and *7*. The overall allocation effect between *Exhibit 7c* and *8c* are the same, only the allocation effects per segment changed and became equal.

In *Exhibit 9 b) and c)* we present the attribution analysis with the drift-allocation effect and drift-interaction effect over 2013 for the equity portfolio that is invested half in developed markets and half in emerging markets that was discussed earlier in *Exhibit 4*. The difference between b) and c) is the total benchmark return which is used in the calculation of the allocation effect. In b) the daily total benchmark return is used. In c) the periodically rebalanced total benchmark return is used. Since the benchmark weight differs over time due to the return that the underlying segments generate, the two returns will differ over time. The total allocation effect does not change, only the distribution of the allocation effect over the different segments changed. In d) we present the one-period attribution results using the original Brinson model. As you can see the multi-period allocation effects of c) (where the daily rebalanced total benchmark is used) are nearly the same as the allocation effect of the one-period analysis. In the table the results are rounded to a basis point, but the difference is 0.25 basis points. *Exhibit 9 c) and d)* mainly differ in the selection effects. The drift-allocation and drift-interaction effect should be allocated to the selection effects, but we couldn't find a mechanism to allocate it correctly to selection effect per segment, so decided to present the drift effects separately.

It can happen that you report over a period where allocation decisions are made at different moments. At the moment that such an allocation decision is made, the passive portfolio weight that is used in the calculation of the drift-allocation and drift-interaction effect should be reset to the portfolio weights as they are at the moment of the allocation decision. If this is not done, some of the allocation effect will get assigned to the drift-allocation effect. The same is the case for the interaction and the drift-interaction effect. When the periodically rebalanced benchmark return is used in the allocation effect (formula 12), the weights that are used to calculate the periodically rebalanced total benchmark return should always be the benchmark weights of the latest allocation decision. Otherwise the distribution of the allocation effect between the different segments will shift.

Exhibit 9

		Allocation Effect	Selection Effect	Interaction Effect	Drift-Allocation Effect	Drift-Interation Effect	Total Effect
a)	**Daily analysis, Cariño smoothed**						
	Developed Markets	0.68%	0.43%	0.04%			1.15%
	Emerging Markets	0.76%	0.38%	-0.05%			1.08%
	Total	**1.43%**	**0.81%**	**-0.02%**			**2.23%**
b)	**Daily analysis, Cariño smoothed, with drift effects**						
	Developed Markets	0.69%	0.43%	0.04%	-0.02%	0.00%	1.15%
	Emerging Markets	0.77%	0.38%	-0.04%	-0.02%	-0.02%	1.08%
	Total	**1.47%**	**0.81%**	**0.00%**	**-0.04%**	**-0.02%**	**2.23%**
c)	**Daily analysis, Cariño smoothed, with drift effects and daily rebalanced total benchmark return used in the allocation effect**						
	Developed Markets	0.73%	0.43%	0.04%	-0.02%	0.00%	1.18%
	Emerging Markets	0.73%	0.38%	-0.04%	-0.02%	-0.02%	1.04%
	Total	**1.47%**	**0.81%**	**0.00%**	**-0.04%**	**-0.02%**	**2.23%**
d)	**Single period analyses**						
	Developed Markets	0.73%	0.44%	0.04%			1.22%
	Emerging Markets	0.73%	0.31%	-0.03%			1.01%
	Total	**1.46%**	**0.75%**	**0.01%**			**2.23%**

Attribution analysis for a portfolio over 2013 containing MSCI Developed and Emerging market investments. The Benchmark is invested 50% in MSCI Developed markets and 50% in MSCI Emerging markets. The portfolio is invested 55% in MSCI Developed Markets excluding Australia and 45% in MSCI Emerging markets including China A Shares. We have used the net USD returns. a) presents the attribution analysis of the daily effects. In b) we have included the drift-Allocation and drift-Interaction effect. c) presents the same daily attribution analysis as in c) only the allocation effect is not calculated using the daily total benchmark return, but with the periodically rebalanced total benchmark return. d) shows the analysis where the year is treated as one period .

Selection effect

With the introduction of the drift-allocation and drift-interaction effects we have managed to provide a more intuitive allocation effect for multi-period Brinson attribution that is comparable to the one-period allocation effect. You can calculate the effects for any sub-period and combine them to a multi period result allowing for reporting over the decisions that were taken independent of the period over which the decisions were taken. We could try to find a similar adjustment for the selection effect, but this proves to be impossible or at least very difficult. We will explain below why it is very difficult and hopefully this might trigger a reader to find a solution.

The selection effect is taking the arithmetic difference between the portfolio and benchmark return and multiplies this with the benchmark weight. To calculate the selection effect for the whole period you would first geometrically link the portfolio and benchmark return separately and then multiply this with the benchmark weight at the start of the period.

For the multi period analyses we could take the arithmetic difference for the sub period and multiply this with the weight of the benchmark at the moment that the last allocation decision was made instead of the weight of the benchmark at the start of the sub period. But when we would combine the selection effects for those sub periods to the overall period we would not get much closer to the selection effect of the overall period since we do not take the geometric compounding effect of only this segment into account.

Two Period Brinson attribution for multiple segments

Until now we have only looked at attribution analyses for two segments. With two segments the allocation decision for one segment automatically determines the allocation for the other, since the sum of the weights should add up to 100%. When you have multiple segments, like countries or industries, the allocation decision is more complex. It is not limited to one decision to overweight or underweight a certain segment, but you also need to decide which other segment(s) will offset this.

The Brinson model does not differentiate between the number of segments in which can be invested so the Brinson formulas 1 until 4 also apply for an analysis with multiple segments. In *Exhibit 10a* we show a one period Brinson analysis for 10 segments. Eight of the ten segments have an allocation bet, only segment 2 and 8 have the same allocation in the portfolio as in the benchmark. Also in 6 of the 10 segments a selection bet was taken, for four segments, segment 6, 7, 9 and 10, the manager decided to hold the benchmark.

There is one segment (9) where the investor has decided to go short and therefore it has a negative weight. The Brinson model does not make any distinction between a segment with a positive or negative weight and the outcome does provide intuitive results. Since the benchmark of segment 9 has a lower return than the overall benchmark the underweighting of that segment was a good decision and added 43 basis points to the total excess return.

In *Exhibit 10b* we have also added a second period for the 10 segment attribution. No additional decisions are made compared to the first period. As you can see an allocation effect arises in the second period for segments 2 and 8, although we have not made any allocation decision for these segments in the first period. The same effect arises which we found in the two segment analysis, where a weight difference between the benchmark and the portfolio occurs at the end of the first period due to the return difference between the portfolio and the benchmark. This return difference is introduced by the selection decisions that are made. The weight difference introduces an allocation effect in the second period which is not due to an active allocation decision, but is a result of analyzing the two periods as separate periods instead of one.

In *Exhibit 11* we present the same analysis, but with the drift-allocation and drift-interaction effect included. As you can see the allocation effect for segments 2 and 8 is not present anymore, but is instead included in the drift-allocation effect. Furthermore, you can see that the allocation decision for segment 1 did not add 19 basis points in the second period, which was suggested by the normal Brinson analysis, but actually added 26 basis points. The 7 basis points difference is due to the drift-allocation which arises in the second period since a selection decision was made in the first period.

If you combine the two period results and smooth the results using the Cariño smoothing as in *Exhibit 11c*, you can see that the allocation results are very similar to the results that you would get if you would perform the analysis over the two periods as if it was one period (*Exhibit 11d*). Other smoothing algorithms like the Menchero or the Frongello will give very similar results.

Exhibit 10

		Portfolio Weight	Benchmark Weight	Portfolio Return	Benchmark Return	Active Return	Allocation effect	Selection Effect	Interaction Effect	Total Effect
a)	**Period 1**									
	Segment 1	31.0%	35.0%	12.00%	8.00%	4.00%	-0.17%	1.40%	-0.16%	1.07%
	Segment 2	25.0%	25.0%	-1.00%	5.00%	-6.00%	0.00%	-1.50%	0.00%	-1.50%
	Segment 3	12.0%	8.0%	4.00%	3.00%	1.00%	-0.03%	0.08%	0.04%	0.09%
	Segment 4	5.0%	9.0%	-1.00%	-2.00%	1.00%	0.23%	0.09%	-0.04%	0.28%
	Segment 5	10.0%	5.0%	0.00%	2.00%	-2.00%	-0.08%	-0.10%	-0.10%	-0.28%
	Segment 6	5.0%	4.3%	-1.00%	-1.00%	0.00%	-0.03%	0.00%	0.00%	-0.03%
	Segment 7	0.0%	4.0%	-4.00%	-4.00%	0.00%	0.31%	0.00%	0.00%	0.31%
	Segment 8	3.5%	3.5%	7.00%	-2.00%	9.00%	0.00%	0.32%	0.00%	0.32%
	Segment 9	-2.0%	3.0%	-5.00%	-5.00%	0.00%	0.43%	0.00%	0.00%	0.43%
	Segment 10	10.5%	3.2%	-5.00%	-5.00%	0.00%	-0.63%	0.00%	0.00%	-0.63%
	Total	**100.0%**	**100.0%**	**3.67%**	**3.63%**	**0.04%**	**0.02%**	**0.29%**	**-0.26%**	**0.04%**
b)	**Period 2**									
	Segment 1	33.5%	36.5%	-3.00%	-8.00%	5.00%	0.19%	1.82%	-0.15%	1.86%
	Segment 2	23.9%	25.3%	1.00%	4.00%	-3.00%	-0.08%	-0.76%	0.04%	-0.80%
	Segment 3	12.0%	8.0%	-4.00%	-2.00%	-2.00%	-0.01%	-0.16%	-0.08%	-0.25%
	Segment 4	4.8%	8.5%	-1.00%	1.00%	-2.00%	-0.10%	-0.17%	0.07%	-0.20%
	Segment 5	9.6%	4.9%	5.00%	3.00%	2.00%	0.23%	0.10%	0.09%	0.42%
	Segment 6	4.8%	4.1%	2.00%	2.00%	0.00%	0.03%	0.00%	0.00%	0.03%
	Segment 7	0.0%	3.7%	-1.00%	-1.00%	0.00%	-0.03%	0.00%	0.00%	-0.03%
	Segment 8	3.6%	3.3%	0.00%	0.50%	-0.50%	0.01%	-0.02%	0.00%	-0.01%
	Segment 9	-1.8%	2.8%	-2.00%	-2.00%	0.00%	0.01%	0.00%	0.00%	0.01%
	Segment 10	9.6%	2.9%	2.00%	2.00%	0.00%	0.25%	0.00%	0.00%	0.25%
	Total	**100.0%**	**100.0%**	**-0.49%**	**-1.77%**	**1.28%**	**0.48%**	**0.82%**	**-0.02%**	**1.28%**
c)	**Two period using Cariño smoothing method**									
	Segment 1	31.0%	35.0%	8.64%	-0.64%	9.28%	0.02%	3.27%	-0.31%	2.98%
	Segment 2	25.0%	25.0%	-0.01%	9.20%	-9.21%	-0.09%	-2.27%	0.05%	-2.31%
	Segment 3	12.0%	8.0%	-0.16%	0.94%	-1.10%	-0.03%	-0.09%	-0.05%	-0.17%
	Segment 4	5.0%	9.0%	-1.99%	-1.02%	-0.97%	0.12%	-0.09%	0.04%	0.07%
	Segment 5	10.0%	5.0%	5.00%	5.06%	-0.06%	0.15%	0.00%	0.00%	0.16%
	Segment 6	5.0%	4.3%	0.98%	0.98%	0.00%	-0.01%	0.00%	0.00%	-0.01%
	Segment 7	0.0%	4.0%	-4.96%	-4.96%	0.00%	0.27%	0.00%	0.00%	0.27%
	Segment 8	3.5%	3.5%	7.00%	-1.51%	8.51%	0.01%	0.29%	0.00%	0.30%
	Segment 9	-2.0%	3.0%	-6.90%	-6.90%	0.00%	0.44%	0.00%	0.00%	0.44%
	Segment 10	10.5%	3.2%	-3.10%	-3.10%	0.00%	-0.36%	0.00%	0.00%	-0.36%
	Total	**41.5%**	**38.2%**	**3.16%**	**1.80%**	**1.37%**	**0.52%**	**1.13%**	**-0.28%**	**1.37%**
d)	**Two period as a single period**									
	Segment 1	31.0%	35.0%	8.64%	-0.64%	9.28%	0.10%	3.25%	-0.37%	2.97%
	Segment 2	25.0%	25.0%	-0.01%	9.20%	-9.21%	0.00%	-2.30%	0.00%	-2.30%
	Segment 3	12.0%	8.0%	-0.16%	0.94%	-1.10%	-0.03%	-0.09%	-0.04%	-0.17%
	Segment 4	5.0%	9.0%	-1.99%	-1.02%	-0.97%	0.11%	-0.09%	0.04%	0.06%
	Segment 5	10.0%	5.0%	5.00%	5.06%	-0.06%	0.16%	0.00%	0.00%	0.16%
	Segment 6	5.0%	4.3%	0.98%	0.98%	0.00%	-0.01%	0.00%	0.00%	-0.01%
	Segment 7	0.0%	4.0%	-4.96%	-4.96%	0.00%	0.27%	0.00%	0.00%	0.27%
	Segment 8	3.5%	3.5%	7.00%	-1.51%	8.51%	0.00%	0.30%	0.00%	0.30%
	Segment 9	-2.0%	3.0%	-6.90%	-6.90%	0.00%	0.43%	0.00%	0.00%	0.43%
	Segment 10	10.5%	3.2%	-3.10%	-3.10%	0.00%	-0.36%	0.00%	0.00%	-0.36%
	Total	**41.5%**	**38.2%**	**3.16%**	**1.80%**	**1.37%**	**0.68%**	**1.07%**	**-0.38%**	**1.37%**

Two-period attribution analysis for 10 segments using the Brinson attribution effects.

Exhibit 11

		Allocation effect	Drift-Allocation effect	Selection Effect	Interaction Effect	Drift-Interaction effect	Total Effect
a)	**Period 1**						
	Segment 1	-0,17%	0,00%	1,40%	-0,16%	0,00%	1,07%
	Segment 2	0,00%	0,00%	-1,50%	0,00%	0,00%	-1,50%
	Segment 3	-0,03%	0,00%	0,08%	0,04%	0,00%	0,09%
	Segment 4	0,23%	0,00%	0,09%	-0,04%	0,00%	0,28%
	Segment 5	-0,08%	0,00%	-0,10%	-0,10%	0,00%	-0,28%
	Segment 6	-0,03%	0,00%	0,00%	0,00%	0,00%	-0,03%
	Segment 7	0,31%	0,00%	0,00%	0,00%	0,00%	0,31%
	Segment 8	0,00%	0,00%	0,32%	0,00%	0,00%	0,32%
	Segment 9	0,43%	0,00%	0,00%	0,00%	0,00%	0,43%
	Segment 10	-0,63%	0,00%	0,00%	0,00%	0,00%	-0,63%
	Total	**0,02%**	**0,00%**	**0,29%**	**-0,26%**	**0,00%**	**0,04%**
b)	**Period 2**						
	Segment 1	0,26%	-0,08%	1,82%	-0,21%	0,06%	1,86%
	Segment 2	0,00%	-0,08%	-0,76%	0,00%	0,04%	-0,80%
	Segment 3	-0,01%	0,00%	-0,16%	-0,08%	0,00%	-0,25%
	Segment 4	-0,10%	0,00%	-0,17%	0,08%	0,00%	-0,19%
	Segment 5	0,23%	-0,01%	0,10%	0,10%	0,00%	0,41%
	Segment 6	0,02%	0,00%	0,00%	0,00%	0,00%	0,02%
	Segment 7	-0,02%	0,00%	0,00%	0,00%	0,00%	-0,02%
	Segment 8	0,00%	0,01%	-0,02%	0,00%	0,00%	-0,01%
	Segment 9	0,02%	0,00%	0,00%	0,00%	0,00%	0,02%
	Segment 10	0,24%	0,00%	0,00%	0,00%	0,00%	0,24%
	Total	**0,64%**	**-0,16%**	**0,82%**	**-0,11%**	**0,09%**	**1,28%**
c)	**Two period using Cariño smoothing method**						
	Segment 1	0,10%	-0,08%	3,27%	-0,37%	0,06%	3,00%
	Segment 2	0,00%	-0,09%	-2,27%	0,00%	0,05%	-2,27%
	Segment 3	-0,04%	0,00%	-0,09%	-0,04%	0,00%	-0,17%
	Segment 4	0,12%	0,00%	-0,09%	0,04%	0,00%	0,07%
	Segment 5	0,16%	-0,01%	0,00%	0,00%	0,00%	0,16%
	Segment 6	-0,01%	0,00%	0,00%	0,00%	0,00%	-0,01%
	Segment 7	0,28%	0,00%	0,00%	0,00%	0,00%	0,28%
	Segment 8	0,00%	0,01%	0,29%	0,00%	0,00%	0,29%
	Segment 9	0,44%	0,00%	0,00%	0,00%	0,00%	0,44%
	Segment 10	-0,37%	0,00%	0,00%	0,00%	0,00%	-0,37%
	Total	**0,68%**	**-0,17%**	**1,13%**	**-0,38%**	**0,10%**	**1,37%**
d)	**Two period as a single period**						
	Segment 1	0,10%	0,00%	3,25%	-0,37%	0,00%	2,97%
	Segment 2	0,00%	0,00%	-2,30%	0,00%	0,00%	-2,30%
	Segment 3	-0,03%	0,00%	-0,09%	-0,04%	0,00%	-0,17%
	Segment 4	0,11%	0,00%	-0,09%	0,04%	0,00%	0,06%
	Segment 5	0,16%	0,00%	0,00%	0,00%	0,00%	0,16%
	Segment 6	-0,01%	0,00%	0,00%	0,00%	0,00%	-0,01%
	Segment 7	0,27%	0,00%	0,00%	0,00%	0,00%	0,27%
	Segment 8	0,00%	0,00%	0,30%	0,00%	0,00%	0,30%
	Segment 9	0,43%	0,00%	0,00%	0,00%	0,00%	0,43%
	Segment 10	-0,36%	0,00%	0,00%	0,00%	0,00%	-0,36%
	Total	**0,68%**	**0,00%**	**1,07%**	**-0,38%**	**0,00%**	**1,37%**

Two-period attribution analyses for 10 segments including the drift-allocation and drift-interaction effect. Also the periodically rebalanced total benchmark return is used in the allocation effect.

Conclusion and discussion

We have shown that the allocation effect in the Brinson model can provide unintuitive results when combining results over multiple periods. An allocation effect can arise for segments where no allocation decision has been made due to the selection decisions in previous periods, independent of the smoothing algorithm used. We have extended the Brinson model by including two effects, the drift-allocation and drift-interaction effect, which capture the effect that arises due to drift of the portfolio and benchmark weights in periods in which no allocation decisions are made[2]. Including those two effects in the multi-period attribution analysis and using a periodically rebalanced total benchmark return in the calculation of the allocation effect will lead to allocation results that become very similar to the allocation effects when evaluating the period as a single period. For a single period analysis the drift-allocation and drift-interaction effects are zero and the model reduces again to the original Brinson model.

The Interaction effect and the new drift-allocation and drift-interaction effects are mainly the result of selection results of earlier periods and, therefore, do not capture an investment decision by itself. There is no separate investment decision to generate excess return in any of those effects. The choice of the length of a period which is used in the measurement of the effects does have an effect on the outcome, but this is a measurement/reporting decision and not an investment decision. Since the attribution analysis evaluates the investment decisions, you could combine the interaction, drift-interaction and drift-allocation effect into one effect when presenting the result of the analysis.

Furthermore we have looked at different (smoothing) methods to get multi-period attribution results that explain the full excess return over the period. One of the methods that we have investigated is the method developed by Berg, which is an example of a CNP method. We have shown that these methods can assign an unexpected selection effect to a segment for which no selection decision was taken. The smoothing algorithms developed by Cariño, Menchero and Frongello, that combine the single period results into results that add up to the multiple period excess return, provide very similar results to each other. All these methods can be used as smoothing algorithm for the defined attribution effects and will present very similar results.

In this article we have concentrated on the Brinson model, but a similar effect will occur when combining the results of other single period additive models over multiple periods. Examples of those models are the Karnoski-Singer [1994] or Geenen, Klok and van de Burgt [2006] multi-currency model and the Van Breukelen fixed income model [2000].

We have shown that allocation effects will be impacted by selection results of earlier subperiods, when there are more subperiods in the calculation then decisions changed. The

[2] The drift-allocation effect measures the impact of the drift of the portfolio weight compared to the portfolio weights when was invested passively (according to benchmark) inside the segments. Although the passive portfolio weight that is used in the calculation of the drift-allocation effect is set to the portfolio weight in the case of a rebalancing, it does not measure the impact of a rebalancing decision. To measure this impact, a reference portfolio and/or benchmark can be set up, which is rebalanced periodically to the original weights. Comparing the result of the real portfolios with those notional portfolios will give you an understanding of the impact of the rebalancing decision.

reader is asked to be aware of these impacts and change their calculation methodology to only measure for periods between changes in the decisions or to add the above presented drift-allocation and drift-interaction effect and implement the allocation effect as presented in formula (12), with the periodically rebalanced total benchmark as reference.

Appendix

In this appendix we will show that a CNP method can provide a selection effect for a segment for which no selection decision is taken. From formula 1 we know that the selection effect for a one-period analysis in the Brinson Model is calculated by:

$$\text{selection effect} = \sum_j w_j^B * \left(r_j^P - r_j^B\right) \quad \text{(A1)}$$

For the selection effect for a segment j for a second period in the CNP method we can write:

$$\text{selection effect for 2nd period} = \sum_j w_{j,2}^B * \left(r_{j,2}^P * \left(1 + R_{TOT,1}^{BMPF}\right) - r_{j,2}^B * \left(1 + R_{TOT,1}^{BMBM}\right)\right) \text{(A2)}$$

Where $R_{TOT,1}^{BMPF} = \sum_j w_{j,1}^B * r_{j,1}^P$ and $R_{TOT,1}^{BMBM} = \sum_j w_{j,1}^B * r_{j,1}^B$.

In the case that no selection decision is taken for segment j the return for the portfolio and the benchmark for that segment is the same and the formula for the selection effect for the 2nd period reduces to:

$$\text{selection effect for 2nd period} = \sum_j w_{j,2}^B * r_{j,2}^B * \left(R_{TOT,1}^{BMPF} - R_{TOT,1}^{BMBM}\right) \quad \text{(A3)}$$

From this formula we can see that there will be a selection effect for the second period for a segment for which no selection decision was taken when the two total returns differ which is the case when at least one of the (other) segments had a selection effect over the first period.

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